MATH 6021 Lecture 11 11/23/2020

Yau Conj ('82): $\exists$ ∞'ly many min. surfaces in ANY closed (M^{n+1}, g).

* Assume, from now on, that $3 \leq n+1 \leq 7$. *

\# All min hypersurf. are closed, smooth & embedded. \#

Thm A: (Marques-Neves '17)

(M^{n+1}, g), $\mathrm{Ric}_g > 0$ (or satisfies "Frankel Property")

$\Rightarrow$ Yau's conj. holds.

Last time ...... using the topology of $\mathcal{Z}_n(M; \mathbb{Z}_2)$, we can make sense of p-sweepouts, this gives the notion of volume spectrum of (M^{n+1}, g), $\{\omega_p(M,g)\}_{p \in \mathbb{N}}$. s.t.

p-width

$$(0 \leq)\ \omega_1 \leq \omega_2 \leq \omega_3 \leq \cdots \leq \boxed{\omega_p} \leq \cdots \quad (\to +\infty)$$

Gromov-Guth: $C_1\, p^{\frac{1}{n+1}} \leq \omega_p \leq C_2\, p^{\frac{1}{n+1}} \quad \forall p$

"Proof of Thm A":

(Analogy: $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{pmatrix}$

E_{λ_1} is 2-dim'l.

modulo scaling $E_{\lambda_1} \sim S^1$)

Case 1: $\omega_p = \omega_{p+1}$ for some p

Lusternik-Schnirelmann theory $\Rightarrow$ $\exists$ ∞'ly many min. hypersurf.

Case 2: $\omega_p < \omega_{p+1}$ for all p.

Argue by contradiction. Suppose NOT, i.e. $\exists$ only finitely many min. hypersurfaces, say $\Sigma_1, \ldots, \Sigma_N$ for some $N \in \mathbb{N}$.

Idea: min-max theory + counting argument

Min-max theory $\Longrightarrow$ $\forall p \in \mathbb{N}$, $\omega_p = \|V_p\|(M)$ for some stationary varifold V_p in M.

s.t. $V_p = n_1^{(p)} \Sigma_1 + n_2^{(p)} \Sigma_2 + \cdots + n_N^{(p)} \Sigma_N$

where $n_i^{(p)} \geq 0$.

Frankel Property $\Longrightarrow$ $V_p = n_{\ell(p)}^{(p)} \Sigma_{\ell(p)}$

- Fix $\delta > 0$ s.t. $\delta < \min_{i=1,\dots,N} \{ \text{Area}(\Sigma_i) \}$. Then,

$$\omega_p = \|V_p\|(M) = n_{\ell(p)}^{(p)} \cdot \text{Area}(\Sigma_{\ell(p)}) > \delta \cdot n_{\ell(p)}^{(p)}$$

$$\Rightarrow \quad n_{\ell(p)}^{(p)} < \frac{\omega_p}{\delta} \underset{\text{Gromov-Guth}}{\leq} \frac{C_2}{\delta} p^{\frac{1}{n+1}}$$

By counting argument, $\forall p \in \mathbb{N}$,

$$\underbrace{p}_{\text{linear in } p} \underset{\text{Case 2}}{=} \# \{ \omega_k : k = 1, \dots, p \} \leq \overbrace{\left(\frac{C_2}{\delta} \cdot N \right) p^{\frac{1}{n+1}}}^{\text{sub-linear in } p}$$

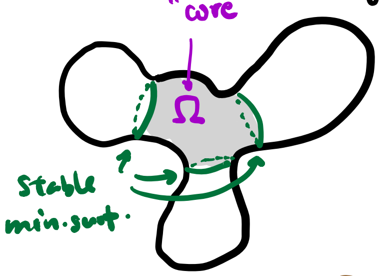
contradiction arise! $\blacksquare$

Song '18 localized their arguments to prove:

Thm B: (Song '18) Yau's conj. holds for ALL (M^{n+1}, g).

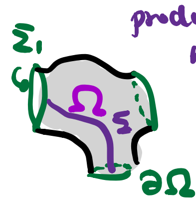
(contradiction)

"Idea of Proof":



truncate ⋯⋯→

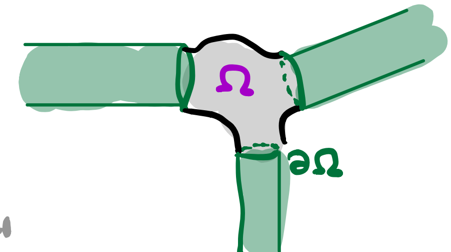
∃ min-max theory for mfd with boundary
↓ (L.-Zhou)
produce free bdy min surf Σ



manifold w/. boundary.

⋯⋯→ cylindrical extension

$(\hat{\Omega}, g)$: non-cpt & not smooth at $\partial\Omega$



$\hat{\Omega} = \Omega \cup (\partial\Omega \times [0, \infty))$

$\hat{g} = g \cup$ "product metric"

The core Ω satisfies Frankel property.

Can still define "cylindrical p-width", $\omega_p(\hat{\Omega}, \hat{g})$, by cpt exhaustion

Key estimate: $\quad p \cdot \text{Area}(\Sigma_1) \leq \omega_p(\hat{\Omega}, \hat{g}) \leq p \cdot \text{Area}(\Sigma_1) + C p^{\frac{1}{n+1}}$

where Σ_1 = component of $\partial\Omega$ with largest area.

- arithmetic lemma $\Rightarrow$ contradiction! $\quad\square$

Weyl Law for the Volume Spectrum

Q: (Gromov) The volume spectrum $\{\omega_p(M,g)\}_{p\in\mathbb{N}}$ satisfies some Weyl Law?

Motivation: (M^{n+1}, g) closed $\rightsquigarrow$ Laplace-Beltrami operator

$$-\Delta : C^\infty(M) \longrightarrow C^\infty(M).$$

Spectrum of $(-\Delta)$: $\underset{\lambda_0}{0} < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow +\infty$

(i.e. $-\Delta f = \lambda f$)

Weyl Law: $\boxed{\lim_{p\to\infty} \lambda_p \cdot p^{-\frac{2}{n+1}} = a(n) \cdot \text{Vol}(M,g)^{-\frac{2}{n+1}}}$

where $a(n)$: explicit dimensional constant.

i.e. $\lambda_p \sim C p^{\frac{2}{n+1}}$ as $p \to \infty$.

Q: How does it relate to min-max theory?

Min-max characterization of λ_p:

Denote: $\quad E(f) := \dfrac{\int_M |\nabla f|^2 dV_g}{\int_M f^2 dV_g}$ $\quad$ "Rayleigh quotient"

(Recall: harmonic function (locally) minimizes $\int |\nabla f|^2$)

$\forall p \in \mathbb{N}$,

$$\lambda_p(M,g) = \inf_{\substack{Q \subset W^{1,2}(M) \\ (p+1)\text{-dim'l subspace}}} \left(\sup_{\substack{f \in Q \\ f \neq 0}} E(f) \right).$$

Observe: $E(cf) = E(f)$ for any constant c

$\overset{\text{descends}}{\rightsquigarrow} \quad E : \mathbb{P}W^{1,2}(M) \longrightarrow \mathbb{R}$

i.e. $$\lambda_p(M,g) = \inf_{\mathbb{RP}^p \subset \mathbb{P}W^{1,2}(M)} \Big(\sup_{[f] \in \mathbb{RP}^p} E(f) \Big).$$

Compare:
$X = \mathbb{RP}^p$
$$\omega_p(M,g) := \inf_{\underbrace{\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)}_{p\text{-sweepout}}} \Big(\sup_{x \in X} \mathbf{M}(\Phi(x)) \Big)$$

The volume spectrum $\{\omega_p(M,g)\}_{p\in\mathbb{N}}$ satisfies a "Weyl Law".

Thm C: (Liokumovich-Marques-Neves '18)

$\exists$ dim'l constant $\alpha(n) > 0$ s.t.
$$\lim_{p\to\infty} \omega_p(M,g) \cdot p^{-\frac{1}{n+1}} = \alpha(n) \cdot \mathrm{Vol}(M,g)^{\frac{n}{n+1}}$$

Remarks: • $\Rightarrow \omega_p \sim C p^{\frac{1}{n+1}}$ as $p \to \infty$ (cf. Gromov-Guth)

• Open Q: compute $\alpha(n)$?

We omit the proof, but look at some consequences.

Recall: Yau Conj $\Rightarrow \exists$ ∞'ly min. hypersurf. in (M^{n+1}, g).

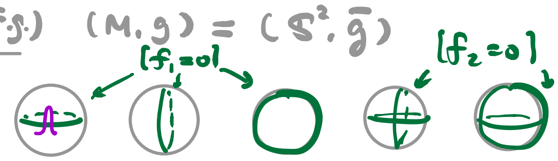
Q: Can we say more (geometry / topology / Morse index) about these min hypersurfaces ?

Partial A: Yes, for generic metric g.

Motivation / Fact: (M^{n+1}, g), Δ-spectrum $\{\lambda_p\} \rightsquigarrow \{f_p\}$ eigenfunctions

$\Rightarrow \{f_p\}_{p\in\mathbb{N}}$ are "equidistributed".

E.g.) $(M,g) = (S^2, \bar{g})$



Thm D: (Irie - Marques - Neves '18)

For generic (M, g), min. hypersurfaces are "dense" in M.

Thm E: (Marques - Neves - Song '18)

For generic (M, g), min. hypersurf. are "equi-distributed" in M,

i.e. $\exists$ seq. $\{\Sigma_j\}_{j \in \mathbb{N}}$ of min. hypersurf. in M s.t. $\forall f \in C^\infty(M)$,

$$\frac{\int_M f \, dV_M}{\mathrm{Vol}(M, g)} = \lim_{q \to \infty} \frac{\sum_{j=1}^{q} \int_{\Sigma_j} f \, dA_{\Sigma_j}}{\sum_{j=1}^{q} \mathrm{Area}(\Sigma_j)}$$

"Sketch of Proof of Thm D": Idea: Weyl Law + perturbation argument.

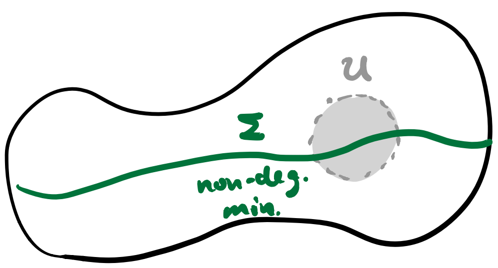
Denote: $\mathcal{M} := \{$ smooth metrics on $M \}$

DH non-deg.

$U \subseteq M$ open

$\mathcal{M}_U := \{ g \in \mathcal{M} : \exists$ non-deg. min. hypersurf. Σ in (M, g) s.t. $\Sigma \cap U \neq \phi \}$, $\mathcal{M}_U \subseteq \mathcal{M}$

(M, g)



$g \in \mathcal{M}_U$.

FACT: $\mathcal{M}_U \subseteq \mathcal{M}$ is open

($\because$ Inverse Function Thm.)

Claim: $\mathcal{M}_U \subseteq \mathcal{M}$ is dense

Thm D follows from Claim by take a countable cover $M = \bigcup_i U_i$.

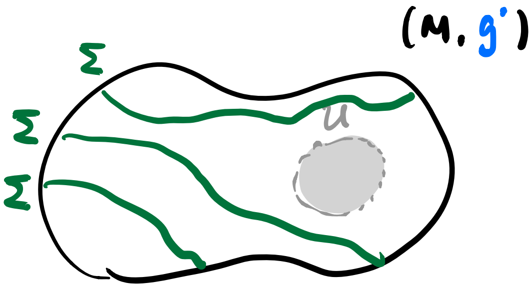
"Proof of Claim": Fix any $g \in \mathcal{M}$.

B. White '91, '17 : Bumpy Metric Thm $\Rightarrow$ $\exists$ g' close to g s.t.

- ALL min hypersurf. Σ in (M, g') is non-deg.

If some $\Sigma \cap U \neq \phi$, then $g' \in \mathcal{M}_U$ $\to$ Done.

Otherwise, ALL min. hypersurf. in (M, g') misses $\mathcal{U}$.



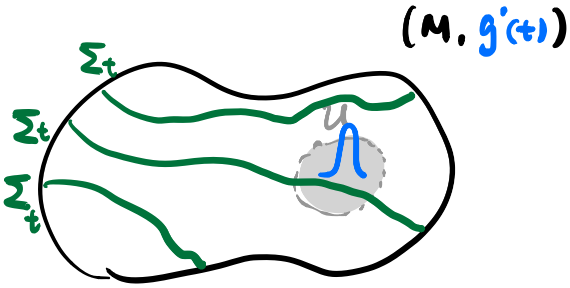
Modify g' inside $\mathcal{U}$ to a 1-parameter of new metrics $g'(t)$, $t \in [0,1]$, st $g'(0) = g'$

s.t. $\mathrm{Vol}(M, g'(t)) > \mathrm{Vol}(M, g')$ $\forall t > 0$.

$\Downarrow$ Weyl Law

$$\boxed{\omega_p(M, g'(t)) > \omega_p(M, g')} \quad (*)$$ for some p

↓ deform the metric in $\mathcal{U}$



Fix this p, and

$t \longmapsto \omega_p(M, g'(t))$ is cts.

(cf. Schoen-Simon-Yau, Schoen-Simon)

B. Sharp '17 : Compactness result for min. hypersurf. with bdd area & index.

$\Rightarrow$ for each $\delta > 0$, then

$$\#\{\Sigma \subset (M, g') \text{ min w/. } \mathrm{Area}(\Sigma) \le \delta,\ \mathrm{index}(\Sigma) \le \delta\} < +\infty$$

$\Rightarrow$ $\left\{ \sum_{j=1}^{N} m_j \,\mathrm{Area}_{g'}(\Sigma_j) \right\}$ is a countable subset of $\mathbb{R}$.

So, $t \longmapsto \omega_p(M, g'(t))$ is in fact constant. contradicts (*).

$\Rightarrow$ Some min hypersurf. in $(M, g'(t))$ must intersect $\mathcal{U}$ for some t.

Apply Bumpy Metric Thm again. $g'(t) \sim g'' \in \mathcal{M}_{\mathcal{U}}$.

$\blacksquare$